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(p, q)-Deformations and (p, q)-Vector Coherent States of the Jaynes-Cummings Model in the Rotating Wave Approximation

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Abstract

Classes of (p, q)-deformations of the Jaynes-Cummings model in the rotating wave approximation are considered. Diagonalization of the Hamiltonian is performed exactly, leading to useful spectral decompositions of a series of relevant operators. The latter include ladder operators acting between adjacent energy eigenstates within two separate infinite discrete towers, except for a singleton state. These ladder operators allow for the construction of (p, q)-deformed vector coherent states. Using (p, q)-arithmetics, explicit and exact solutions to the associated moment problem are displayed, providing new classes of coherent states for such models. Finally, in the limit of decoupled spin sectors, our analysis translates into (p, q)-deformations of the supersymmetric harmonic oscillator, such that the two supersymmetric sectors get intertwined through the action of the ladder operators as well as in the associated coherent states.

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1 Introduction

In recent years, quantum algebras and groups [1] which appear as a generalization of the symmetry concept [2] and the basics of so-called noncommutative theories, have been the subject of intensive research interest in both mathematics and physics. The q- and more generally (p, q)-deformation of a pre-defined algebraic structure [3, 4, 5] proves to be a powerful tool widely used in the representation theory of quantum groups. The field of "q-mathematics" has a long history [6, 7] dating back to over 150 years, and includes several famous names such as Cauchy, Jacobi and Heine to mention just a few. Its possible relation to physics has been considerably reinforced during the last thirty years [3, 8]. In particular, great attention has been devoted to deformations of the bosonic Fock-Heisenberg algebra. The most commonly studied deformed bosons, with annihilation and creation operators a and $a^\dagger$, respectively, satisfy the q-commutation relation [3] (also called quommutation)

$$aa^\dagger - qa^\dagger a = \mathbb{I}, \tag{1}$$

or some variant forms of such a relation [4, 9]. Still more general deformations, which include in specific limits the above standard q-deformed case and which also provides consistent extensions of the harmonic oscillator algebra, proceed from the two parameter deformation of the Fock algebra introduced by Chakrabarty and Jagannathan [5], namely the so-called (p, q)-oscillator quantum algebras generated by three operators a, $a^\dagger$ and N which obey [5, 10]

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad aa^\dagger - qa^\dagger a = p^{-N}, \quad aa^\dagger - p^{-1}a^\dagger a = q^N. \tag{2}$$

Here, p and q are free parameters, which henceforth are chosen to be both real and such that $p > 1$, $0 < q < 1$ and $pq < 1$. Clearly, one recovers the ordinary Fock algebra of the harmonic oscillator algebra in the double limit $p, q \to 1$, with then $[a, a^\dagger] = \mathbb{I}$ and $N = a^\dagger a$. Furthermore, these q- and (p, q)-deformed algebras have found a number of relevant applications and provide algebraic interpretations of various q- and (p, q)-special functions [9, 10, 11].

The harmonic oscillator algebra is central in the construction of a number of models in physics, among which the Jaynes–Cummings model ($\mathcal{JC}m$) plays a significant role. Indeed ever since Jaynes and Cummings' historical work [12], the $\mathcal{JC}m$ has been at the basis of many investigations. This system belongs to a class of physically relevant models widely used in atomic physics and quantum optics. As far as we know, a great deal of analytically solvable models of this type have been studied in the rotating wave approximation (r.w.a.) within the framework of non-deformed commutative theories (see [12]–[17] and references therein). The $\mathcal{JC}m$ has also been considered in the context of generalized intensity dependent oscillator algebras including nonlinear dynamical supersymmetry [18] or using shape invariance techniques [19, 20]. Comparatively, much fewer papers have dealt with generalizations of these models including deformations. Among the latter and mainly based on the generalized intensity-dependent coupling of Buck and Sukumar [21], one may mention, on the one hand, the work by Chaichan *et al.* [22], and on the other hand, that by Chang [23], both dealing with a generalized q-deformed intensity-dependent interaction Hamiltonian of the $\mathcal{JC}m$ given by the Holstein-Primakoff $su_q(1, 1)$ or $su_q(2)$ quantum algebra realizations of the Hamiltonian field operators and the related Peremolov, Glauber or Barut-Girardello group theoretical construction of coherent states. In the same vein, the paper by Naderi *et al.* considers the dynamical properties of a two-level atom in three variants of the two-photon q-deformed $\mathcal{JC}m$ [24]. In this latter work, the authors focused their attention onto the time evolution of atomic properties including population inversion and quantum fluctuations of the atomic dipole variables. However, it is not clear to us how the main issues related to the moment problem as well as the mathematical foundation of the coherent and squeezed states which they use and on which a great part of their analysis rests in a crucial way, are solved.

In a recent publication [14], Hussin and Nieto have performed an interesting systematic search of different types of ladder operators for the $\mathcal{JC}m$ model in the r.w.a. and constructed associated coherent states. In the present work, and in line with that investigation, we provide a generalization of that analysis to (p,q)-deformations of the same model.

The outline of the paper is the following. In Section 2, we briefly recall the main results relevant to the $\mathcal{JC}m$ in the r.w.a. in the non-deformed situation [14]. Section 3 then introduces (p,q)-deformations of the same model. By providing an explicit diagonalization of the (p,q)-deformed Hamiltonian, the spectrum and its eigenstates are exactly identified. As in the non-deformed case [14], except for a singleton state, all other energy eigenstates are organized into two separate discrete towers, for which ladder operators transforming states into one another within each tower separately may be introduced. Using properties of these ladder operators, in Section 4 we introduce general classes of (p,q)-deformed vector coherent states. The freedom afforded in their construction is fixed from two alternative points of view, discussed in Section 5, which in the ordinary case of the non-deformed Fock algebra coincide. However at all stages of our discussion, the double limit $p,q \to 1$ reproduces the corresponding results of [14]. Section 5 also briefly considers the situation in the uncoupled limit of the $\mathcal{JC}m$, while Section 6 presents some concluding remarks. An Appendix collects useful facts in connection with properties of (p,q)-deformed algebras and related functions.

2 The Ordinary $\mathcal{JC}m$ in the Rotating Wave Approximation

The $\mathcal{JC}m$ describes the interaction between one mode of the quantized electromagnetic field and a two-level model of an atomic system [12, 14]–[16]. It has proved to be a theoretical laboratory of great relevance to many topics in atomic physics and quantum optics, as well as in the study of ion traps, cavity QED theory and quantum information processing [13, 14]. Furthermore, the spin-orbit interaction term which appears in the $\mathcal{JC}m$ is essentially the so-called Dresselhaus spin-orbit term [25]. The model is thus also widely used in condensed matter physics for its relevance in spintronics [26] which exploits the electron spin rather than its charge to develop a new generation of electronic devices [27, 28]. The solution of the complete $\mathcal{JC}m$ is not yet known in a closed form [14]. However, in the r.w.a., although the Hamiltonian remains nonlinear, the model becomes exactly solvable in closed form with explicit expressions for its eigenenergy states. In this Section, we briefly recall, in a streamlined presentation, the main results in the non-deformed case (see [14, 15] and references therein) of relevance to our analysis of (p,q)-deformations hereafter.

In the r.w.a., the reduced dimensionless $\mathcal{JC}m$ Hamiltonian reads [15]

$$\mathcal{H}^{\rm red} = \frac{1}{\hbar\omega_0}\mathcal{H} = (1+\epsilon)\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{2}\sigma_3 + \lambda\left(a^\dagger\sigma_- + a\sigma_+\right), \tag{3}$$

where a and $a^\dagger$ are the usual photon annihilation and creation operators, respectively, obeying the ordinary Fock algebra, and $(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices with $\sigma_\pm = \sigma_1 \pm i\sigma_2$. The r.w.a. is related to the detuning parameter ϵ which is such that $|\epsilon| \ll 1$, with ω_0 being the fixed atomic frequency and $\omega = \omega_0(1+\epsilon)$ the actual field mode frequency. The r.w.a. is reliable provided $|\omega - \omega_0| \ll \omega, \omega_0$. Finally, λ is the reduced spin-orbit coupling modelling the interaction strength between the radiation field and the atom.

The Hilbert space $\mathcal{V}$ of the system is the tensor product of the Fock space representation of the Fock algebra $(a, a^\dagger)$ and the 2-dimensional representation of the SU(2) algebra associated to the Pauli matrices. A basis of the former is provided by the number operator, $N = a^\dagger a$, orthonormalized eigenstates $|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle$ $(n = 0, 1, 2, \cdots)$, with $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $N|n\rangle = n|n\rangle$, while a basis of the latter spin sector is the orthonormalized set $\{|+\rangle, |-\rangle\}$ such that $\sigma_3|\pm\rangle = \pm|\pm\rangle$. The tensor product space is thus spanned by the states $|n, \pm\rangle = |n\rangle \otimes |\pm\rangle$.

The diagonalization of the Hamiltonian (3) is readily achieved. The orthonormalized energy eigenspectrum consists of a "singleton" state $|E_*\rangle$,

$$\mathcal{H}^{\rm red}|E_*\rangle = E_*|E_*\rangle, \tag{4}$$

with

$$E_* = \frac{1}{2}\epsilon, \qquad |E_*\rangle = |0, -\rangle, \tag{5}$$

and two infinite discrete towers of states $|E_n^\pm\rangle$ such that $\mathcal{H}^{\rm red}|E_n^\pm\rangle = E_n^\pm|E_n^\pm\rangle$ for all $n = 0, 1, 2, \cdots$, expressed as [14]

$$|E_n^+\rangle = \sin\vartheta(n)\,|n, +\rangle + \cos\vartheta(n)\,|n+1, -\rangle, \tag{6}$$

$$|E_n^-\rangle = \cos\vartheta(n)\,|n, +\rangle - \sin\vartheta(n)\,|n+1, -\rangle, \tag{7}$$

where, given $Q(n+1) = \sqrt{\epsilon^2/4 + \lambda^2(n+1)}$, the mixing angle $\vartheta(n)$ is such that

$$\sin\vartheta(n) = {\rm sign}(\lambda)\sqrt{\frac{Q(n+1) - \epsilon/2}{2Q(n+1)}}, \qquad \cos\vartheta(n) = \sqrt{\frac{Q(n+1) + \epsilon/2}{2Q(n+1)}}, \tag{8}$$

while the energy eigenvalues are

$$E_n^\pm = (1+\epsilon)(n+1) \pm Q(n+1). \tag{9}$$

Consequently, one has the spectral decomposition of the reduced Hamiltonian (3),

$$\mathcal{H}^{\rm red} = |E_*\rangle\, E_*\, \langle E_*| \;+\; \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle\, E_n^\pm\, \langle E_n^\pm|. \tag{10}$$

It proves useful to introduce the following notations. Let $\mathcal{V}_0$ be the (complex) one-dimensional subspace of the Hilbert space $\mathcal{V}$ spanned by the state $|0, -\rangle = |E_*\rangle$, and $\overline{\mathcal{V}}$ be its complement in the Hilbert space $\mathcal{V}$, spanned by $\{|E_n^\pm\rangle, n \in \mathbb{N}\}$. We thus have $\mathcal{V} = \mathcal{V}_0 \oplus \overline{\mathcal{V}}$.

Furthermore let us introduce [14] operators $\mathcal{U}$ and $\mathcal{U}^\dagger$ defined through their action on the above two sets of basis vectors, for all $n \in \mathbb{N}$,

$$\mathcal{U}|n, \pm\rangle = |E_n^\pm\rangle; \qquad \mathcal{U}^\dagger|E_*\rangle = 0, \quad \mathcal{U}^\dagger|E_n^\pm\rangle = |n, \pm\rangle, \tag{11}$$

namely

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle\langle n, \pm|, \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle\langle E_n^\pm|. \tag{12}$$

Clearly we have

$$\mathcal{U}\,\mathcal{V} = \overline{\mathcal{V}}; \qquad \mathcal{U}^\dagger\,\mathcal{V} = \mathcal{V}, \quad \mathcal{U}^\dagger\,\overline{\mathcal{V}} = \mathcal{V}. \tag{13}$$

Note that even though neither $\mathcal{U}$ nor $\mathcal{U}^\dagger$ is unitary on the full Hilbert space $\mathcal{V}$, they are the adjoint of one another, hence the notation.

It is of interest to apply these operators onto the quantum Hamiltonian (3). One obtains

$$\mathbb{H}^{\rm red} = \mathcal{U}^\dagger\, \mathcal{H}^{\rm red}\, \mathcal{U} = \sum_{n=0,\pm}^{\infty} |n,\pm\rangle\, E_n^\pm\, \langle n,\pm|, \tag{14}$$

and conversely,

$$\mathcal{U}\, \mathbb{H}^{\rm red}\, \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle\, E_n^\pm\, \langle E_n^\pm| = \mathcal{H}^{\rm red}\, -\, |E_*\rangle\, E_*\, \langle E_*|. \tag{15}$$

The energy eigenstates spanning $\overline{\mathcal{V}}$ may be organized into two subspaces referred to as "towers", namely $\{|E_n^+\rangle, n\in\mathbb{N}\}$ and $\{|E_n^-\rangle, n\in\mathbb{N}\}$. The states in the tower $\{|E_n^+\rangle, n\in\mathbb{N}\}$ are associated to strictly increasing eigenvalues so that they constitute a nondegenerate set of eigenstates. The second group does not necessarily possess the same feature depending on the values for the parameters λ and ϵ. It is possible [16] to identify a range of values for these parameters such that $\{|E_n^-\rangle, n\in\mathbb{N}\}$ only contains nondegenerate states of strictly increasing eigenvalues with n. Some of the considerations discussed hereafter may require a nondegenerate spectrum, which may always be achieved by properly "detuning" the parameters λ and ϵ away from a degenerate case, but not necessarily a strictly increasing spectrum in the label $n\in\mathbb{N}$. Whatever the case may be though, bounded from below spectra such that $E_n^\pm > E_0^\pm$ for $n=1,2,\cdots$ are always assumed implicitly.

It is possible to consider ladder operators acting between successive energy eigenstates within each of the above two towers, irrespective of whether the spectral values are strictly increasing or not[1]. Namely, let us first consider operators $\mathbb{M}^-$ and $\mathbb{M}^+$ given as

$$\mathbb{M}^- = \sum_{n=0,\pm}^{\infty} |n-1,\pm\rangle\, K_\pm(n)\, \langle n,\pm|; \qquad \mathbb{M}^+ = \sum_{n=0,\pm}^{\infty} |n+1,\pm\rangle\, K_\pm^*(n+1)\, \langle n,\pm|, \tag{16}$$

where $K_\pm(n)$ are, at this stage, arbitrary complex coefficients such that $K_\pm(0)=0$. Then, introduce the ladder operators

$$\mathcal{M}^- = \mathcal{U}\, \mathbb{M}^-\, \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |E_{n-1}^\pm\rangle\, K_\pm(n)\, \langle E_n^\pm|; \quad \mathcal{M}^+ = \mathcal{U}\, \mathbb{M}^+\, \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |E_{n+1}^\pm\rangle\, K_\pm^*(n+1)\, \langle E_n^\pm|, \tag{17}$$

which are thus such that, for all $n=0,1,2,\cdots$,

$$\mathcal{M}^-|E_*\rangle = 0, \quad \mathcal{M}^-|E_n^\pm\rangle = K_\pm(n)|E_{n-1}^\pm\rangle; \quad \mathcal{M}^+|E_*\rangle = 0, \quad \mathcal{M}^+|E_n^\pm\rangle = K_\pm^*(n+1)|E_{n+1}^\pm\rangle. \tag{18}$$

Note that $\mathcal{M}^-$ and $\mathcal{M}^+$ are adjoint of one another but in effect only act on the subspace $\overline{\mathcal{V}}$.

General vector coherent states (VCS) may then be introduced [29]–[32] on the space $\overline{\mathcal{V}}$ as eigenstates of the lowering operator $\mathcal{M}^-$ with as eigenvalue an arbitrary complex number $z\in\mathbb{C}$. Furthermore, these VCS are also parametrized by two real quantities $\tau_\pm$ which account for their stability under time evolution generated by the operator $\exp\left\{-i\omega_0 t\, \mathcal{H}^{\rm red}\right\}$, as well as the two spherical coordinates $(\theta,\phi)\in[0,\pi]\times[0,2\pi[$ parametrizing a unit vector in the 2-sphere

[1]We differ on this point with [14], where strictly increasing energy spectra in each tower are required.

S_2 (hence the name of "vector" coherent states). Explicitly, one has [14]

$$
\begin{aligned}
|z;\tau_\pm;\theta,\phi\rangle \quad = \quad & N^+(|z|)\cos\theta\sum_{n=0}^{\infty}\frac{z^n}{K_+(n)!}e^{-i\omega_0\tau_+E_n^+}\,|E_n^+\rangle \\
& + N^-(|z|)\,e^{i\phi}\sin\theta\sum_{n=0}^{\infty}\frac{z^n}{K_-(n)!}e^{-i\omega_0\tau_-E_n^-}\,|E_n^-\rangle,
\end{aligned}
\tag{19}
$$

where $K_\pm(n)! = \prod_{k=1}^{n} K_\pm(k)$ (with, by convention, $K_\pm(0)! = 1$), while the normalization factors are defined as

$$
N^\pm(|z|) = \left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{|K_\pm(n)!|^2}\right]^{-1/2}
\tag{20}
$$

in order that the VCS be of unit norm. The smallest value, R, of the two convergence radii of these two series in $|z|$ also defines the disk D_R in $z \in \mathbb{C}$ for which these VCS are well defined. These states are clearly such that

$$
\mathcal{M}^-|z;\tau_\pm;\theta,\phi\rangle = z\,|z;\tau_\pm;\theta,\phi\rangle, \quad e^{-i\omega_0 t\,\mathcal{H}^{\rm red}}\,|z;\tau_\pm;\theta,\phi\rangle = |z;t+\tau_\pm;\theta,\phi\rangle.
\tag{21}
$$

Further restrictions are necessary to finally specify in a unique fashion the factors $K_\pm(n)$, and then solve the moment problem implied by the requirement of overcompleteness over $\overline{\mathcal{V}}$ for the VCS (19) given a choice of a SU(2) matrix-valued integration measure over $\mathbb{C}\times S_2$ [30]-[32]. Different choices are available [14], each leading to a different set of VCS. Furthermore, taking the limit case $\lambda \to 0$ or the zero-detuning limit (resonance case) $\epsilon \to 0$, different models arise with their associated VCS.

For the sake of illustration, let us consider one such choice explicitly [14]. The factors $K_\pm(n)$ may be restricted for example by requiring that the ladder operators $\mathcal{M}^-$ and $\mathcal{M}^+$ obey the usual Fock algebra of annihilation and creation operators on the space $\overline{\mathcal{V}}$,

$$
\left[\mathcal{M}^-,\mathcal{M}^+\right] = \mathcal{M}^-\,\mathcal{M}^+ \,-\, \mathcal{M}^+\,\mathcal{M}^- \,=\, \mathbb{I}_{\overline{\mathcal{V}}} = \sum_{n=0,\pm}^{\infty}|E_n^\pm\rangle\,\langle E_n^\pm|.
\tag{22}
$$

From the expressions in (18) and the initial conditions $K_\pm(0) = 0$, it follows that the quantities $K_\pm(n)$ are now determined up to arbitrary phase factors $\varphi_\pm(n)$ as

$$
K_\pm(n) = e^{i\varphi_\pm(n)}\,\sqrt{n}, \qquad n = 0,1,2,\cdots.
\tag{23}
$$

Consequently, one has $N^\pm(|z|) = e^{-|z|^2/2}$, which is well-defined for all $z \in \mathbb{C}$. Hence so are then all the VCS $|z;\tau_\pm;\theta,\phi\rangle$.

3 The (p,q)-Deformed $\mathcal{JC}m$ in the Rotating Wave Approximation

Let us now introduce a (p,q)-deformation of the $\mathcal{JC}m$ Hamiltonian (3), namely (p,q)-$\mathcal{JC}m$ models. The eigenstates and spectrum are first identified, before considering the construction of ladder operators following the same rationale as in Section 2. A study of the associated VCS and examples of exactly solvable reduced models is differed to Section 4.

3.1 Energy spectrum and eigenstates

Given the (p,q)-deformation (2) of the ordinary Fock algebra (see the Appendix for further details and identities pertaining to such deformations), we now consider (p,q)-deformations of the Hamiltonian (3) of the form[2]

$$\mathcal{H}^{red} = (1+\epsilon)\left\{h(p,q)[N]+\frac{1}{2}\right\}+\frac{1}{2}\sigma_3+\lambda\left(a^\dagger\sigma_- + a\sigma_+\right), \tag{24}$$

where $[N]=(p^{-N}-q^N)/(p^{-1}-q)$, and $h(p,q)$ is some arbitrary positive function of the real parameters $p>1$ and $0<q<1$ (with $pq<1$) such that $\lim_{p,q\to 1} h(p,q)=1$ in order to recover (3) in the non-deformed case.

The Hilbert space $\mathcal{V}$ of quantum states of the model is again the tensor product of the (p,q)-deformed Fock space spanned by the states[3] $|n\rangle$ ($n\in\mathbb{N}$) such as $a|n\rangle=\sqrt{[n]}|n-1\rangle$ and $a^\dagger|n\rangle=\sqrt{[n+1]}|n+1\rangle$ (see the Appendix), with the 2-dimensional representation of the SU(2) algebra associated to the Pauli matrices σ_i ($i=1,2,3$). Hence the diagonalization of (24) is readily achieved in the same way as in the non-deformed case, on the basis $|n,\pm\rangle=|n\rangle\otimes|\pm\rangle$ of $\mathcal{V}$.

For any $n\in\mathbb{N}$, let us introduce the following quantities,

$$\mathcal{E}([n+1]) = (1+\epsilon)\,h(p,q)\Big([n+1]-[n]\Big)-1,\quad Q([n+1])=\sqrt{\frac{1}{4}\mathcal{E}^2([n+1])\,+\,\lambda^2\,[n+1]}, \tag{25}$$

as well as the mixing angles $\vartheta([n])$ defined by

$$\sin\vartheta([n])=\mathrm{sign}(\lambda)\sqrt{\frac{Q([n+1])-\mathcal{E}([n+1])/2}{2Q([n+1])}},\quad \cos\vartheta([n])=\sqrt{\frac{Q([n+1])+\mathcal{E}([n+1])/2}{2Q([n+1])}}. \tag{26}$$

The energy eigenspectrum of (24) is then obtained as follows. First, there exists a singleton state $|E_*\rangle=|0,-\rangle$ such that

$$\mathcal{H}^{\rm red}\,|E_*\rangle = E_*\,|E_*\rangle,\qquad E_*=\frac{1}{2}\epsilon, \tag{27}$$

with an eigenvalue which is thus independent of the deformation parameters p and q. Next, one also finds two infinite discrete towers of states for all $n\in\mathbb{N}$ such that

$$|E_n^+\rangle \;=\; \sin\vartheta([n])\,|n,+\rangle\,+\,\cos\vartheta([n])\,|n+1,-\rangle, \tag{28}$$

$$|E_n^-\rangle \;=\; \cos\vartheta([n])\,|n,+\rangle\,-\,\sin\vartheta([n])\,|n+1,-\rangle, \tag{29}$$

with

$$\mathcal{H}^{\rm red}\,|E_n^\pm\rangle = E_n^\pm\,|E_n^\pm\rangle,\quad E_n^\pm=\frac{1}{2}(1+\epsilon)\left\{h(p,q)\Big([n+1]+[n]\Big)+1\right\}\pm Q([n+1]). \tag{30}$$

Note that the energy spectrum of these states is deformed by the parameters p and q as compared to the ordinary case. In particular, the Zeeman spin splitting $\Delta E_n=E_n^+-E_n^-=2Q([n+1])$,

[2]Make no mistake that henceforth, all quantities correspond to the (p,q)-deformed analysis even though the notations used coincide with those of Section 2 and do not make explicit the fact that all expressions correspond now to the deformed case. When wanting to make the difference explicit, notations such as for instance $[N]\equiv[N]_{(p,q)}=(p^{-N}-q^N)/(p^{-1}-q)$ and $[n]\equiv[n]_{(p,q)}=(p^{-n}-q^n)/(p^{-1}-q)$ are used.

[3]Once again, the states $|n\rangle=|n\rangle_{(p,q)}$ are not to be confused with the number operator eigenstates of the ordinary Fock algebra as in Section 2, in spite of an identical notation.

proportional to the Rabi frequency, is function of the values for p and q. In terms of these results, the reduced Hamiltonian (24) possesses the spectral resolution

$$\mathcal{H}^{\rm red} = |E_*\rangle\, E_*\, \langle E_*| \; + \sum_{n=0,\pm}^{\infty} |E_n^{\pm}\rangle\, E_n^{\pm}\, \langle E_n^{\pm}|. \tag{31}$$

Let us again introduce the following notations and operators. Let $\mathcal{V}_0$ denote the subspace of the Hilbert space $\mathcal{V}$ spanned by the singleton state $|E_*\rangle = |0,-\rangle$, and $\overline{\mathcal{V}}$ its complement in $\mathcal{V}$, namely the subspace spanned by $\{|E_n^{\pm}\rangle, n \in \mathbb{N}\}$, with of course $\mathcal{V} = \mathcal{V}_0 \oplus \overline{\mathcal{V}}$. Acting on these spaces, let us consider the operators

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} |E_n^{\pm}\rangle\, \langle n, \pm|; \qquad \mathcal{U}^{\dagger} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle\, \langle E_n^{\pm}|, \tag{32}$$

such that, for all $n = 0, 1, 2, \cdots$,

$$\mathcal{U}|n, \pm\rangle = |E_n^{\pm}\rangle; \quad \mathcal{U}^{\dagger}|E_*\rangle = 0, \quad \mathcal{U}^{\dagger}|E_n^{\pm}\rangle = |n, \pm\rangle, \tag{33}$$

and thus

$$\mathcal{U}\,\mathcal{V} = \overline{\mathcal{V}}; \quad \mathcal{U}^{\dagger}\,\mathcal{V} = \mathcal{V}, \quad \mathcal{U}^{\dagger}\,\overline{\mathcal{V}} = \mathcal{V}. \tag{34}$$

Hence once again the operators $\mathcal{U}$ and $\mathcal{U}^{\dagger}$, even though non unitary on $\mathcal{V}$, are adjoint of one another. More specifically, one has

$$\mathcal{U}^{\dagger}\,\mathcal{U} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle\, \langle n, \pm| = \mathbb{I}_{\mathcal{V}}, \qquad \mathcal{U}\,\mathcal{U}^{\dagger} = \sum_{n=0,\pm}^{\infty} |E_n^{\pm}\rangle\, \langle E_n^{\pm}| = \mathbb{I}_{\overline{\mathcal{V}}}. \tag{35}$$

Applying these operators to the reduced Hamiltonian, one finds

$$\mathbb{H}^{\rm red} = \mathcal{U}^{\dagger}\, \mathcal{H}^{\rm red}\, \mathcal{U} = \sum_{n=0,\pm}^{\infty} |n, \pm\rangle\, E_n^{\pm}\, \langle n, \pm|, \tag{36}$$

and conversely,

$$\mathcal{U}\, \mathbb{H}^{\rm red}\, \mathcal{U}^{\dagger} = \sum_{n=0,\pm}^{\infty} |E_n^{\pm}\rangle\, E_n^{\pm}\, \langle E_n^{\pm}| = \mathcal{H}^{\rm red} \; - \; |E_*\rangle\, E_*\, \langle E_*|. \tag{37}$$

Some remarks on the spectrum are in order. First, as in the ordinary $\mathcal{JC}m$, except for the singleton state $|E_*\rangle = |0, -\rangle$, the spectrum is the direct sum of two towers of states $\{|E_n^{\pm}\rangle, n \in \mathbb{N}\}$. However, in contradistinction to the non-deformed case or even the q-deformation with $p = 1$, the (p, q)-basic numbers $[n] = [n]_{(p,q)}$ are not strictly increasing as a function of $n \in \mathbb{N}$ when $p > 1$, $0 < q < 1$ and $pq < 1$. There always exists a finite positive value $n_0 \in \mathbb{N}$ such that $[n]$ decreases once $n > n_0$. Hence, depending on the values for the parameters λ and ϵ as well as the positive function $h(p, q)$, parts of the spectrum $E_n^{\pm}$ may turn negative or present some degeneracies (as in [16]). Without exploring this issue any further in the present work, henceforth we shall assume that parameter values are such that no degeneracies occur and that the spectrum E_n^- remains bounded from below (E_n^+ is obviously positive). The definition of the ladder operators to be considered next does not require a strictly increasing spectrum, while it is only for one of possible choices leading to vector coherent states to be discussed hereafter that the condition of non degeneracy in $E_n^{\pm} > E_0^{\pm}$, for $n \geq 1$, becomes relevant. Since it has been

shown [16] that such conditions may be met in the non-deformed case for appropriate ranges of values for the available parameters, through an argument of continuity in the deformation parameters p and q, similar ranges ought to exist also for the (p, q)-deformed realizations of the $\mathcal{JC}m$ model.

Another feature of potential interest related to these facts, and which will also not be pursued here, is the possibility that through the (p, q)-deformation of the $\mathcal{JC}m$, the levels E_n^+ and E_{n+1}^- cross one another. Such a property may lead to effects similar to the phenomenon of resonant spin-Hall conductance at the Fermi level recently observed in spintronics [27, 28]. Note that this (p, q)-dependent crossing phenomenon is expected since the Zeeman splitting ΔE_n is also modified as a function of p and q. This remark is also in line with the recent suggestion [33, 34, 35] that (p, q)-deformed or space noncommutative realizations of exactly solvable systems may provide useful model approximations to more realistic complex interacting dynamics of collective phenomena.

3.2 Ladder operators

In order to construct ladder operators mapping each of the successive states $|E_n^\pm\rangle$ into one another separately within each of the towers, let us first introduce the following operators acting on $\mathcal{V}$,

$$\mathbb{A}^- = \sum_{n=0,\pm}^{\infty} |n-1,\pm\rangle\, K_\pm([n])\, \langle n,\pm|; \quad \mathbb{A}^+ = \sum_{n=0,\pm}^{\infty} |n+1,\pm\rangle\, K_\pm^*([n+1])\, \langle n,\pm|, \tag{38}$$

where $K_\pm([n])$ are arbitrary complex quantities such that $K_\pm([0]) = K_\pm(0) = 0$. Note that $\mathbb{A}^-$ and $\mathbb{A}^+$ are adjoint of one another on $\mathcal{V}$.

Then the relevant ladder operators are obtained as

$$\mathcal{A}^- = \mathcal{U}\,\mathbb{A}^-\,\mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |E_{n-1}^\pm\rangle\, K_\pm([n])\, \langle E_n^\pm|; \quad \mathcal{A}^+ = \mathcal{U}\,\mathbb{A}^+\,\mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |E_{n+1}^\pm\rangle\, K_\pm^*([n+1])\, \langle E_n^\pm|. \tag{39}$$

Consequently, we have indeed, for all $n \in \mathbb{N}$,

$$\mathcal{A}^-|E_*\rangle = 0, \quad \mathcal{A}^-|E_n^\pm\rangle = K_\pm([n])\,|E_{n-1}^\pm\rangle; \qquad \mathcal{A}^+|E_*\rangle = 0, \quad \mathcal{A}^+|E_n^\pm\rangle = K_\pm^*([n+1])\,|E_{n+1}^\pm\rangle. \tag{40}$$

Note that $\mathcal{A}^-$ and $\mathcal{A}^+$ are adjoint of one another, but that in effect they act only on the subspace $\overline{\mathcal{V}}$.

It is of course possible to express these ladder operators in the $|n,\pm\rangle$ basis. In the case of the lowering operator, one finds

$$\begin{aligned} \mathcal{A}^- \;=\; & \textstyle\sum_{n=0}^{\infty} |n,+\rangle\, \mathcal{A}_{++}^-(n)\, \langle n+1,+| \;+\; \sum_{n=0}^{\infty} |n,+\rangle\, \mathcal{A}_{+-}^-(n)\, \langle n+2,-| \\ & + \textstyle\sum_{n=0}^{\infty} |n,-\rangle\, \mathcal{A}_{-+}^-(n)\, \langle n,+| \;+\; \sum_{n=0}^{\infty} |n,-\rangle\, \mathcal{A}_{--}^-(n)\, \langle n+1,-| \end{aligned} \tag{41}$$

where

$$\mathcal{A}_{++}^-(n) = \sin\vartheta([n])\, \sin\vartheta([n+1])\, K_+([n+1]) + \cos\vartheta([n])\, \cos\vartheta([n+1])\, K_-([n+1]),$$

$$\mathcal{A}_{+-}^-(n) = \sin\vartheta([n])\, \cos\vartheta([n+1])\, K_+([n+1]) - \cos\vartheta([n])\, \sin\vartheta([n+1])\, K_-([n+1]),$$

$$
\begin{aligned}
\mathcal{A}^-_{-+}(n) &= \cos\vartheta([n-1])\,\sin\vartheta([n])\,K_+([n]) \;-\; \sin\vartheta([n-1])\,\cos\vartheta([n])\,K_-([n]), \\
\mathcal{A}^-_{--}(n) &= \cos\vartheta([n-1])\,\cos\vartheta([n])\,K_+([n]) \;+\; \sin\vartheta([n-1])\,\sin\vartheta([n])\,K_-([n]).
\end{aligned} \tag{42}
$$

Likewise for the raising operator,

$$
\begin{aligned}
\mathcal{A}^+ \;=\; & \textstyle\sum_{n=0}^{\infty} |n+1,+\rangle\,\left(\mathcal{A}^-_{++}(n)\right)^*\,\langle n,+| \;+\; \sum_{n=0}^{\infty} |n,+\rangle\,\left(\mathcal{A}^-_{-+}(n)\right)^*\,\langle n,-| \\
& + \textstyle\sum_{n=0}^{\infty} |n+2,-\rangle\,\left(\mathcal{A}^-_{+-}(n)\right)^*\,\langle n,+| \;+\; \sum_{n=0}^{\infty} |n+1,-\rangle\,\left(\mathcal{A}^-_{--}(n)\right)^*\,\langle n,-|.
\end{aligned} \tag{43}
$$

Note that we have $\mathcal{A}^-_{-+}(0) = 0 = \mathcal{A}^-_{--}(0)$, since $K_\pm([0]) = 0$.

The quantities $K_\pm([n])$ parametrize the freedom available in the choice of such ladder operators. Further restrictions arise when considering first the possible existence of vector coherent states meeting a series of general conditions charateristic of such states [30]-[32], starting with one involving the lowering operator $\mathcal{A}^-$ itself.

4 (p,q)-Vector Coherent States for the (p,q)-$\mathcal{JC}m$

By considering the action of the lowering operator $\mathcal{A}^-$, we are able to construct an overcomplete set of vectors in $\overline{\mathcal{V}}$, so-called vector coherent states [30]-[32] for the (p,q)-$\mathcal{JC}m$. Since these states are associated to unit vectors in the 2-sphere S_2 [29], they are referred to as (p,q)-vector coherent states ((p,q)-VCS). As in Section 2, these (p,q)-VCS are parametrized by a complex variable $z \in \mathbb{C}$, two real parameters $\tau_\pm$ to track a stable time evolution of the (p,q)-VCS, and finally the spherical angle coordinates (θ,ϕ) on S_2, $|z;\tau_\pm;\theta,\phi\rangle$. In the double limit that $p,q \to 1$, these (p,q)-VCS reduce to those of [14] discussed in Section 2. The dependence of the (p,q)-VCS on all these quantities is introduced as follows, according to the discussion in [30].

4.1 Identifying (p,q)-VCS

As a slight extension of the analysis so far, given two real parameters μ and ν, let us consider the operator

$$
\mathbb{Q}_{\mathcal{V}} = |E_*\rangle\,\langle E_*| \;+\; \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle \left(\frac{q^\mu}{p^\nu}\right)^n \langle E_n^\pm|. \tag{44}
$$

Hence, the energy eigenstates of the (p,q)-$\mathcal{JC}m$ are also eigenstates of this operator $\mathbb{Q}_{\mathcal{V}}$, with eigenvalues given through the above spectral decomposition.

We are now in a position to successively identify the dependence of the (p,q)-VCS to be constructed on each of the parameters of which they are functions, first z, then $\tau_\pm$, and finally, θ and ϕ. Having defined both the operators $\mathcal{A}^-$ and $\mathbb{Q}_{\mathcal{V}}$, let us consider the following eigenvalue problem in z for the (p,q)-VCS,

$$
\mathcal{A}^-|z;\tau_\pm;\theta,\phi\rangle = z\,\mathbb{Q}_{\mathcal{V}}\,|z;\tau_\pm;\theta,\phi\rangle \tag{45}
$$

which generalizes to a two-level system the definition of coherent states as advocated in [30]-[32]. The particular case $\mu = 0 = \nu$ yields also a consistent definition of (p,q)-VCS viewed as the limit $\mu,\nu \to 0$ of the present definition (note that their domain of definition in z, required

for the convergence of the infinite series to be considered hereafter, may have to be adapted accordingly).

By expanding the (p,q)-VCS in the Hamiltonian eigenstate basis as

$$|z;\tau_\pm;\theta,\phi\rangle = C_*(z)|E_*\rangle + \sum_{n=0,\pm}^{\infty} C_n^\pm(z)|E_n^\pm\rangle, \tag{46}$$

where $C_*(z)$ and $C_n^\pm(z)$ are complex continuous functions of z to be specified presently, the condition (45) then requires, for all $n \in \mathbb{N}$,

$$C_*(z) = 0, \qquad C_{n+1}^\pm(z) K_\pm([n+1]) = z\,\frac{q^{\mu n}}{p^{\nu n}}\,C_n^\pm(z), \tag{47}$$

of which the solution is

$$C_n^\pm(z) = \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)/2} \frac{z^n}{K_\pm([n])!}\,C_0^\pm(z), \tag{48}$$

where $C_0^\pm(z)$ are arbitrary complex functions of z, while we defined $K_\pm([n])! = \prod_{k=1}^n K_\pm([k])$ with, by convention, $K_\pm([0])! = 1$. Hence, the general solution to (45) defines states lying only within the subspace $\overline{\mathcal{V}}$, of the form

$$|z;\tau_\pm;\theta,\phi\rangle = \sum_{n=0,\pm}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)/2} \frac{z^n}{K_\pm([n])!}\,C_0^\pm(z)\,|E_n^\pm\rangle. \tag{49}$$

Note that the eigenvalue problem (45) is singular at the particular value $z=0$, since its solution is an arbitrary superposition of the three states $|E_*\rangle$ and $|E_0^\pm\rangle$. Nevertheless, we shall consider the (p,q)-VCS associated to $z=0$, $|z=0;\tau_\pm;\theta,\phi\rangle$, as being defined through the continuous limit in $z \to 0$ of the construction in (49), namely $|z=0;\tau_\pm;\theta,\phi\rangle = C_0^+(0)|E_0^+\rangle + C_0^-(0)|E_0^-\rangle$.

Let us now turn to the issue of the stability of the (p,q)-VCS under time evolution generated by the Hamiltonian (24). Namely, we now require furthermore that (p,q)-VCS are transformed into one another under time evolution according to the following dependence on the real parameters $\tau_\pm$, for all $t \in \mathbb{R}$,

$$e^{-i\omega_0 t\,\mathcal{H}^{\rm red}}\,|z;\tau_\pm;\theta,\phi\rangle = |z;t+\tau_\pm;\theta,\phi\rangle. \tag{50}$$

Since one has, for all $n \in \mathbb{N}$,

$$e^{-i\omega_0 t\,\mathcal{H}^{\rm red}}\,|E_n^\pm\rangle = e^{-i\omega_0 t\,E_n^\pm}\,|E_n^\pm\rangle, \tag{51}$$

one needs to factor out their complex phases from the quantities $K_\pm([n])$,

$$K_\pm([n]) = e^{i\varphi_\pm([n])} K_\pm^0([n]), \tag{52}$$

where $K_\pm^0([n]) > 0$ are now real positive scalars. The stability condition (50) is then solved by choosing, for all $n = 1, 2, \cdots$,

$$\varphi_\pm([n]) = \omega_0\tau_\pm\left[E_n^\pm - E_{n-1}^\pm\right], \tag{53}$$

and redefining

$$C_0^\pm(z) = \mathcal{C}_0^\pm(z)\,e^{-i\omega_0\tau_\pm E_0^\pm}, \tag{54}$$

where $\mathcal{C}_0^\pm(z)$ are new complex functions of z. Hence,

$$|z;\tau_\pm;\theta,\phi\rangle = \sum_{n=0,\pm}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)/2} \frac{z^n}{K_\pm^0([n])!}\,\mathcal{C}_0^\pm(z)\, e^{-i\omega_0\tau_\pm E_n^\pm}\,|E_n^\pm\rangle. \tag{55}$$

Having identified both the z and $\tau_\pm$ dependences of the coherent states, finally let us account for their (θ,ϕ) dependence and S_2 vector character implicit so far through the two functions $\mathcal{C}_0^\pm(z)$. The latter are now chosen to be given as

$$\mathcal{C}_0^+(z) = N^+(|z|)\,\cos\theta, \qquad \mathcal{C}_0^-(z) = N^-(|z|)\,e^{i\phi}\,\sin\theta, \tag{56}$$

$N^\pm(|z|)$ being factors such that the constructed (p,q)-VCS be of unit norm,

$$N^\pm(|z|) = \left\{\sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{\left(K_\pm^0([n])!\right)^2}\right\}^{-1/2}. \tag{57}$$

The convergence radii $R_\pm$ of these two series in z,

$$R_\pm = \lim_{n\to\infty}\left\{(q^\mu p^{-\nu})^{-(n-1)}\,K_\pm^0([n])\right\}, \tag{58}$$

depend on the choice of functions $K_\pm^0([n])$ as well as on (μ,ν) possibly. Specific cases are considered hereafter.

Consequently, the (p,q)-VCS constructed here are properly defined provided $z \in D_R$ where D_R denotes the disk in the complex plane centered at $z=0$ and of radius $R = \min(R_+,R_-)$. Their general structure is thus of the form

$$\begin{aligned} |z;\tau_\pm;\theta,\phi\rangle \quad = \quad & N^+(|z|)\,\cos\theta \sum_{n=0}^{\infty}\left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)/2} \frac{z^n}{K_+^0([n])!}\,e^{-i\omega_0\tau_+ E_n^+}\,|E_n^+\rangle \\ & + N^-(|z|)\,e^{i\phi}\,\sin\theta \sum_{n=0}^{\infty}\left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)/2} \frac{z^n}{K_-^0([n])!}\,e^{-i\omega_0\tau_- E_n^-}\,|E_n^-\rangle. \end{aligned} \tag{59}$$

Only the real positive functions $K_\pm^0([n])$ still need to be specified. They parametrize the remaining freedom in the construction. Particular examples will be considered hereafter by imposing further requirements on these (p,q)-VCS. Note that the double limit $p,q\to 1$ yields the VCS of the non-deformed $\mathcal{JC}m$ as obtained by Hussin and Nieto [14], briefly described in Section 2.

4.2 Some expectation values

Before dealing with further requirements on the family of (p,q)-VCS, among which their overcompleteness in the space $\overline{\mathcal{V}}$, let us consider some relevant expectation values for these states. Given (59), the mean value of $\mathcal{H}^{\rm red}$ for any of the (p,q)-VCS is simply

$$\begin{aligned} \langle\mathcal{H}^{\rm red}\rangle \quad = \quad & |N^+(|z|)|^2\,\cos^2\theta \sum_{n=0}^{\infty}\left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{\left(K_+^0([n])!\right)^2}\,E_n^+ \\ & + |N^-(|z|)|^2\,\sin^2\theta \sum_{n=0}^{\infty}\left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{\left(K_-^0([n])!\right)^2}\,E_n^-. \end{aligned} \tag{60}$$

Likewise for the "number" operator associated to the ladder operators $\mathcal{A}^-$ and $\mathcal{A}^+$, one finds the expectation value

$$\begin{aligned}\langle \mathcal{A}^+ \mathcal{A}^- \rangle &= |z|^2 \Bigg\{ \; |N^+(|z|)|^2 \cos^2\theta \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n+1)} \frac{|z|^{2n}}{\left(K_+^0([n])!\right)^2} \\ &\qquad + |N^-(|z|)|^2 \sin^2\theta \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n+1)} \frac{|z|^{2n}}{\left(K_-^0([n])!\right)^2} \Bigg\}. \end{aligned} \tag{61}$$

Finally, the average atomic spin time evolution $\langle \sigma_3(t) \rangle = \langle U^{-1}(t) \sigma_3 U(t) \rangle$, with $U(t) = exp\{-i\omega_0 t\, \mathcal{H}^{\rm red}\}$ being the time evolution operator, has the form

$$\begin{aligned}\langle \sigma_3(t) \rangle &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} |z|^{2n} \frac{\mathcal{E}([n+1])}{Q([n+1])} \left\{ -\frac{|N^+(|z|)|^2}{\left(K_+^0([n])!\right)^2} \cos^2\theta + \frac{|N^-(|z|)|^2}{\left(K_-^0([n])!\right)^2} \sin^2\theta \right\} \\ &+ \lambda N^+(|z|)\, N^-(|z|) \sin 2\theta \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{K_+^0([n])! K_-^0([n])!} \frac{[n+1]}{Q([n+1])} \cos \Psi_n(t), \end{aligned} \tag{62}$$

with

$$\Psi_n(t) = \omega_0 \left[(t+\tau_+)\, E_n^+ \;-\; (t+\tau_-)\, E_n^- \right] \;+\; \phi = \omega_0 \Delta E_n\, t \;+\; \omega_0 \left[\tau_+ E_n^+ - \tau_- E_n^- \right] + \phi. \tag{63}$$

As is the case in the non-deformed model, the explicit time dependence which arises for the atomic inversion $\langle \sigma_3(t) \rangle$ is due to the mixed state sector, namely the fact that the mixed-spin matrix elements of the Heisenberg picture operator $\sigma_3(t)$ do not vanish when $\lambda \neq 0$. Hence, the proposition which states that the time dependence of atomic inversion consists of Rabi oscillations when a system is prepared in a coherent state of the radiation field [17] extends to (p, q)-VCS. However, in the limit where $\lambda \to 0$, no such oscillations occur. Let us also point out that the time dependence of $\langle \sigma_3(t) \rangle$ diplays chaotic behaviour for appropriate values of the model parameters, as was previously mentioned for the q-deformation of the model, with $0 < q < 1$, in the work by Naderi *et al.* [24].

4.3 Overcompleteness and the moment problem

An important property that coherent states ought to meet is that of overcompleteness in the space over which they are defined [30]. In the present case, this means that the (p, q)-VCS in (59) must also provide a resolution of the identity operator over the subspace $\overline{\mathcal{V}}$, namely

$$\mathbb{I}_{\mathcal{V}} = \mathbb{I}_{\mathcal{V}_0} \;+\; \mathbb{I}_{\overline{\mathcal{V}}} = |E_*\rangle\, \langle E_*| \;+\; \mathbb{I}_{\overline{\mathcal{V}}}, \tag{64}$$

while

$$\mathbb{I}_{\overline{\mathcal{V}}} = \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle\, \langle E_n^\pm| = \int_{D_R \times S_2} d\mu(z;\theta,\phi)\, |z;\tau_\pm;\theta,\phi\rangle\, \langle z;\tau_\pm;\theta,\phi|, \tag{65}$$

where $d\mu(z;\theta,\phi)$ is some SU(2) matrix-valued integration measure over $D_R \times S_2$ to be determined from the above requirement.

Let us thus consider the following parametrization of that measure,

$$d\mu(z;\theta,\phi) = d^2z\, d\theta\, \sin\theta\, d\phi \left\{ \mathcal{W}^+(|z|) \sum_{n=0}^{\infty} |E_n^+\rangle\langle E_n^+| \;+\; \mathcal{W}^-(|z|) \sum_{n=0}^{\infty} |E_n^-\rangle\langle E_n^-| \right\}, \tag{66}$$

in terms of real weight functions $\mathcal{W}^{\pm}(|z|)$ to be identified. Using the radial parametrization $z = r\, e^{i\varphi}$ and $d^2z = dr\, r\, d\varphi$ where $r \in [0, \infty[$ and $\varphi \in [0, 2\pi[$, a direct substitution in (65) leads to the moment problem associated to the overcompleteness relation (65). In terms of the functions $h^{\pm}(r^2)$ defined through

$$h^{+}(r^2) = \frac{4\pi^2}{3}\, |N^{+}(r)|^2\, \mathcal{W}^{+}(r), \qquad h^{-}(r^2) = \frac{8\pi^2}{3}\, |N^{-}(r)|^2\, \mathcal{W}^{-}(r), \tag{67}$$

the following two infinite sets of moment identities must be met, for all $n \in \mathbb{N}$,

$$\int_0^{R^2} du\, u^n\, h^{\pm}(u) = \left(\frac{q^{\mu}}{p^{\nu}}\right)^{-n(n-1)} \left(K^0_{\pm}([n])!\right)^2. \tag{68}$$

In conclusion, the resolution of the identity operator over $\overline{\mathcal{V}}$ in terms of the (p, q)-VCS is achieved provided the Stieljes moment problem (68) can be solved [36, 37]. This requires a choice of functions $K^0_{\pm}([n]) > 0$ such that not only the conditions (68) may all be met, but also such that the normalization factors $N^{\pm}(|z|)$ converge in a non-empty disc of the complex plane.

As a result of this analysis, *a priori* there may exist a large number of sets of (p, q)-VCS which fulfill all the above properties, namely continuity in the complex parameter z, temporal stability through a simple additive time dependence in the real parameters $\tau_{\pm}$, a unit vector valued characterization on the sphere S_2 in terms of the spherical coordinates θ and ϕ, and the completeness property of a resolution of the unit operator with a SU(2) matrix-valued integration measure over these spaces. These sets of (p, q)-VCS are distinguished from one another by different choices of real positive weight factors $K^0_{\pm}([n])$, in agreement with the considerations developed in [30, 38]. The above construction of (p, q)-VCS is general, but can admit explicit exact solutions to the moment problem (68) for particular cases. Concrete examples are discussed in Section 5..

4.4 Action-angle variables

One of the useful properties that general coherent states constructed according to the arguments of [38] possess, is that action-angle variables are readily identified in relation to the continuous parameters ensuring stability of the coherent states under time evolution. In the present case, canonical reduced action-angle variables $(J_{\pm}(t), \tau_{\pm}(t))$ are such that for the previously evaluated expectation values of the reduced Hamiltonian (24) in the (p, q)-VCS, one has

$$\langle \mathcal{H}^{\rm red} \rangle = J_{+}\, \omega_{+} \ + \ J_{-}\, \omega_{-} = \sum_{\pm} J_{\pm}\, \omega_{\pm}, \tag{69}$$

in relation to the action-angle variational principle of the form

$$\int dt \sum_{\pm} \left[\frac{d\tau_{\pm}}{dt}\, J_{\pm} \ - \ \omega_{\pm}\, J_{\pm}\right] \ \longleftrightarrow \ \int dt \left[\langle \frac{i}{\omega_0} \frac{d}{dt} \rangle \ - \ \langle \mathcal{H}^{\rm red} \rangle\right], \tag{70}$$

where $\omega_{\pm}$ are two constant factors to be chosen appropriately. Consequently

$$\frac{d\tau_{\pm}}{dt} = \frac{\partial \langle \mathcal{H}^{\rm red} \rangle}{\partial J_{\pm}} = \omega_{\pm}, \qquad \frac{dJ_{\pm}}{dt} = -\frac{\partial \langle \mathcal{H}^{\rm red} \rangle}{\partial \tau_{\pm}} = 0. \tag{71}$$

Given the time evolution, $\tau_\pm(t) = t + \tau_\pm(0)$, one simply finds $\omega_\pm = 1$. From the expression in (60), one then has the identifications

$$\begin{aligned} J_+ &= |N^+(|z|)|^2 \cos^2\theta \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{\left(K_+^0([n])!\right)^2} E_n^+ , \\ J_- &= |N^-(|z|)|^2 \sin^2\theta \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{\left(K_-^0([n])!\right)^2} E_n^- . \end{aligned} \tag{72}$$

As a final remark, let us mention that the saturated Heisenberg uncertainty relations which are obeyed by q- and (p,q)-coherent states are also well-known in q-mechanics (see for instance [39]). Such minimal uncertainties may be characterized through small corrections to canonical commutation relations defined in [39, 40]. Such properties in the case of the (p,q)-VCS constructed here are deferred to a later study.

5 Explicit Solutions

In order to completely specify the quantities $K_\pm^0([n])$, one last set of conditions needs to be implemented. In the present Section, two such choices are discussed, one of which allows for an exact and explicit solution to the moment problem, hence the construction of a set of (p,q)-VCS. First, in line with the illustrative example of Section 2, we consider restricting the algebra of the ladder operators $\mathcal{A}^\pm$. Then as a second and independent possibility, we apply a final additional criterion developed in [30] in order to uniquely characterize a set of coherent states which meet already all the requirements considered heretofore and having led to the representation (59), even though the moment problem remains unsolved for that choice.

5.1 Constraining the ladder operator algebra

In order to uniquely identify the set of functions $K_\pm^0([n]) > 0$, let us consider the possibility that this may be achieved by restricting the algebraic properties of the ladder operators. In line with the general (p,q)-deformations of the Fock algebra in (2), let us constrain the algebra of the operators $\mathbb{A}^\pm$ acting on $\mathcal{V}$ to be such that

$$\begin{aligned} \mathbb{A}^-\,\mathbb{A}^+ - q_0\,\mathbb{A}^+\,\mathbb{A}^- = p_0^{-N} &= \sum_{n=0,\pm}^{\infty} |n,\pm\rangle\, p_0^{-n}\,\langle n,\pm| , \\ \mathbb{A}^-\,\mathbb{A}^+ - p_0^{-1}\,\mathbb{A}^+\,\mathbb{A}^- = q_0^{N} &= \sum_{n=0,\pm}^{\infty} |n,\pm\rangle\, q_0^{n}\,\langle n,\pm| , \end{aligned} \tag{73}$$

where p_0 and q_0 are again two real parameters such that $p_0 > 1$, $0 < q_0 < 1$ and $p_0 q_0 < 1$, which may or may not be identical to p and q. For instance, we could have $p_0 = 1$ and $q_0 = 1$ thus corresponding to an ordinary Fock algebra, or else $p_0 = p$ and $q_0 = q$, but also more generally $p_0 = p^\alpha$ and $q_0 = q^\alpha$, α being some real constant. As a matter of fact, exact solutions to the moment problem are presented hereafter in all these situations.

In terms of the ladder operators $\mathcal{A}^\pm = \mathcal{U}\,\mathbb{A}^\pm\,\mathcal{U}^\dagger$ acting on the subspace $\overline{\mathcal{V}}$, the associated algebraic constraint reads

$$\mathcal{A}^-\,\mathcal{A}^+ - q_0\,\mathcal{A}^+\,\mathcal{A}^- = \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle\, p_0^{-n}\,\langle E_n^\pm| ,$$

$$\mathcal{A}^- \mathcal{A}^+ - p_0^{-1} \mathcal{A}^+ \mathcal{A}^- = \sum_{n=0,\pm}^{\infty} |E_n^\pm\rangle \, q_0^n \, \langle E_n^\pm|. \tag{74}$$

Whether in terms of (73) or (74), these algebraic constraints translate into the following identities, for all $n \in \mathbb{N}$,

$$\left(K_\pm^0([n+1])\right)^2 - q_0 \left(K_\pm^0([n])\right)^2 = p_0^{-n}, \quad \left(K_\pm^0([n+1])\right)^2 - p_0^{-1} \left(K_\pm^0([n])\right)^2 = q_0^n. \tag{75}$$

Given the initial values $K_\pm^0([0]) = 0$, the solution to these recursion relations is simply

$$K_\pm^0([n]) = \sqrt{[n]_{(p_0,q_0)}} = \sqrt{[n]_{(q_0^{-1},p_0^{-1})}}, \tag{76}$$

where[4]

$$[n]_{(p_0,q_0)} = \frac{p_0^{-n} - q_0^n}{p_0^{-1} - q_0} = \frac{\left(q_0^{-1}\right)^{-n} - \left(p_0^{-1}\right)^n}{\left(q_0^{-1}\right)^{-1} - \left(p_0^{-1}\right)} = [n]_{(q_0^{-1},p_0^{-1})}. \tag{77}$$

Given this solution, the normalization factors are defined by the series

$$|N^\pm(|z|)|^{-2} = \sum_{n=0}^{\infty} \left(\frac{q^\mu}{p^\nu}\right)^{n(n-1)} \frac{|z|^{2n}}{[n]_{(p_0,q_0)}!}, \tag{78}$$

of which the convergence radius is

$$R = \lim_{n\to\infty} \left[\left(\frac{q^\mu}{p^\nu}\right)^{-2(n-1)} \frac{p_0^{-n} - q_0^n}{p_0^{-1} - q_0}\right]^{1/2} = \lim_{n\to\infty} \left[\left(p_0 p^{-2\nu} q^{2\mu}\right)^{-(n-1)} \frac{1 - (p_0 q_0)^n}{1 - (p_0 q_0)}\right]^{1/2}. \tag{79}$$

Provided $p_0 p^{-2\nu} q^{2\mu} < 1$, a condition which we shall henceforth assume to be satisfied[5], this radius of convergence is infinite, $R = \infty$, and the moment problem (68) then becomes, for all $n \in \mathbb{N}$,

$$\int_0^\infty du\, u^n \, h^\pm(u) = \left(\frac{q^\mu}{p^\nu}\right)^{-n(n-1)} \left([n]_{(p_0,q_0)}!\right). \tag{80}$$

In order to solve these equations, the Ramanujan integral (121) discussed in the Appendix suggests itself quite naturally, through a simple but appropriate rescaling of its arguments in the form of (123).

After a little moment's thought one comes to the conclusion that a solution to (80) based on (123) is possible for the following choice of parameters,

$$\mu = \frac{1}{2}, \qquad \nu = 0, \qquad p_0 = p, \qquad q_0 = q, \tag{81}$$

in which case $p_0 p^{-2\nu} q^{2\mu} = pq < 1$, hence corresponding indeed to an infinite radius of convergence. For this choice, one has (for definitions of the (p,q)-exponential functions appearing in these expressions, see the Appendix),

$$h^\pm\left(|z|^2\right) = \frac{\left(p^{-1} - q\right)}{q \log\left(1/pq\right)} \, e_{(p,q)} \left(-|z|^2 \, p^{-1/2} q^{-1} \left(p^{-1} - q\right)\right), \tag{82}$$

[4]Incidentally, it is because of this identity, corresponding to the exchange $p_0 \leftrightarrow q_0^{-1}$, that the two solutions to the above two recursion relations are consistent, as are the two algebraic restrictions in (73) and (74).

[5]If $p_0 p^{-2\nu} q^{2\mu} = 1$, the radius of convergence is finite with $R = (1 - p_0 q_0)^{-1/2}$, while when $p_0 p^{-2\nu} q^{2\mu} > 1$ the radius of convergence vanishes, implying that (p,q)-VCS cannot be constructed in such a case.

as well as[6]

$$\left(K^0_\pm([n])\right)^2 = [n], \qquad |N^\pm(|z|)|^{-2} = \mathcal{E}^{(1/2,0)}_{(p,q)}\left(|z|^2 q^{-1/2}\left(p^{-1}-q\right)\right), \tag{83}$$

with for the weight functions $\mathcal{W}^\pm(|z|)$ in the integration measure (66) of the overcompleteness relation (65),

$$\mathcal{W}^+(|z|) = \frac{3}{4\pi^2}\,|N^+(|z|)|^{-2}\,h^+\left(|z|^2\right), \qquad \mathcal{W}^-(|z|) = \frac{3}{8\pi^2}\,|N^-(|z|)|^{-2}\,h^-\left(|z|^2\right). \tag{84}$$

Explicit expressions for all previously computed quantities readily follow, beginning with the definition of the associated (p,q)-VCS which then meet all the necessary requirements expected of coherent states. Note that up to the coefficients $3/(2\pi)$ and $3/(4\pi)$, the reduced weights obtained are compatible with that of the q-shape invariant harmonic oscillator [20]. Furthermore, (82) is a (p,q)-generalization of the q-harmonic oscillator coherent state moment problem solution constructed in [41]. Finally, in the double limit $p,q\to 1$, the results of [14] are recovered.

The functions (82) thus provide a complete and explicit solution to the moment problem of the (p,q)-VCS for the (p,q)-$\mathcal{JC}m$ such that the ladder operators $\mathcal{A}^\pm$ obey the same (p,q)-Fock algebra as the original modes a and $a^\dagger$ of the initial Hamiltonian (24), namely with the choice $p_0=p$ and $q_0=q$. It is also possible to construct an explicit solution when the ladder operators $\mathcal{A}^\pm$ are constrained to rather obey the ordinary non-deformed Fock algebra on $\overline{\mathcal{V}}$, corresponding to the choice $p_0=1$ and $q_0=1$. One then has to consider[7], for all $n\in\mathbb{N}$,

$$K^0_\pm([n]) = \sqrt{n}, \qquad \int_0^\infty du\, u^n\, h^\pm(u) = \left(\frac{q^\mu}{p^\nu}\right)^{-n(n-1)}(n!)\,, \qquad p^{-\nu}q^\mu \le 1. \tag{85}$$

An obvious solution to this moment problem is obtained when $\mu=0=\nu$, in which case the condition for an infinite radius of convergence is saturated. One then has

$$h^\pm\left(|z|^2\right) = e^{-|z|^2}, \qquad |N^\pm(|z|)|^{-2} = e^{|z|^2}, \qquad \mathcal{W}^+(|z|) = \frac{3}{4\pi^2}, \qquad \mathcal{W}^-(|z|) = \frac{3}{8\pi^2}. \tag{86}$$

In fact, the above two explicit solutions belong to a general class of solutions obtained by taking $(p_0,q_0)=(p^\alpha,q^\alpha)$ with α a positive real parameter, $\alpha>0$, such that $p^{\alpha-2\nu}q^{2\mu}<1$ in order to ensure an infinite radius of convergence[8] in $z\in\mathbb{C}$. Once again based on (123), an explicit solution to the moment problem (80) is achieved for the following choice of parameters,

$$\mu = \frac{1}{2}\alpha, \qquad \nu = 0, \qquad p_0 = p^\alpha, \qquad q_0 = q^\alpha, \tag{87}$$

for which the radius of convergence is indeed infinite, $p^{\alpha-2\nu}q^{2\mu}=(pq)^\alpha<1$. One then has

$$h^\pm\left(|z|^2\right) = \frac{(p^{-\alpha}-q^\alpha)}{q^\alpha\log\left(1/p^\alpha q^\alpha\right)}\,e_{(p^\alpha,q^\alpha)}\left(-|z|^2\,p^{-\alpha/2}q^{-\alpha}\left(p^{-\alpha}-q^\alpha\right)\right), \tag{88}$$

with

$$|N^\pm(|z|)|^{-2} = \mathcal{E}^{(1/2,0)}_{(p^\alpha,q^\alpha)}\left(|z|^2 q^{-\alpha/2}\left(p^{-\alpha}-q^\alpha\right)\right), \tag{89}$$

leading finally to the weight functions $\mathcal{W}^\pm(|z|)$ given in terms of the latter two quantities through the same relations as in (84). In the limits that $\alpha\to 1$ or $\alpha\to 0$, the previous two explicit solutions are then recovered as particular cases.

[6]Restricting to $p_0=p$ and $q_0=q$ but keeping μ and ν arbitrary such that $p^{1-2\nu}q^{2\mu}<1$ in order to retain an infinite radius of convergence, one has $\left(K^0_\pm([n])\right)^2=[n]$ and $|N^\pm(|z|)|^{-2}=\mathcal{E}^{(\mu,\nu)}_{(p,q)}\left(|z|^2\,p^\nu\,q^{-\mu}\left(p^{-1}-q\right)\right)$, hence also all other previous expressions given accordingly.

[7]Leading to $|N^\pm(|z|)|^{-2}=\mathfrak{e}^{(\mu,\nu)}_{(p,q)}\left(|z|^2\,p^\nu\,q^{-\mu}\right)$, which converges for all $|z|<\infty$ provided $p^{-\nu}q^\mu\le 1$.

[8]Leading to $|N^\pm(|z|)|^{-2}=\mathcal{E}^{(\mu/\alpha,\nu/\alpha)}_{(p^\alpha,q^\alpha)}\left(|z|^2p^\nu q^{-\mu}(p^{-\alpha}-q^\alpha)\right)$.

5.2 The action identity constraint

An alternative to fixing the factors $K^0_\pm([n])$ through conditions on the algebra of ladder operators, is to consider the action identity constraint discussed in [30] as the one last requirement which singles out coherent states uniquely. In the case of the ordinary Fock algebra, this action identity constraint is equivalent to requiring that the ladder operators obey themselves the Fock algebra as well. We shall establish that this is not the case for the (p,q)-VCS of the (p,q)-$\mathcal{JC}m$ constructed above.

Given the relations (72), in the present model the action identity constraint is of the form

$$J_+ = \cos^2\theta\left(|z|^2 + E^+_0\right), \qquad J_- = \sin^2\theta\left(|z|^2 + E^-_0\right). \tag{90}$$

By direct substitution into these constraints of the relations (72), the identification of the successive powers in $|z|^2$ leads to the following solution for the factors $K^0_\pm([n])$,

$$K^0_\pm([n]) = \left(\frac{q^\mu}{p^\nu}\right)^{(n-1)} \sqrt{E^\pm_n - E^\pm_0}. \tag{91}$$

These positive real quantities are thus well-defined provided one has $E^\pm_n > E^\pm_0$ for all $n \geq 1$, as is implicitly assumed. It is noteworthy that, as $(p,q) \to (1^+, 1^-)$, these factors reduce to exactly those obtained in [16] by the factorization method. On the other hand, since the present solution for $K^0_\pm([n])$ cannot be brought into the form of (76) for some choice of constants p_0 and q_0 meeting our assumptions for these quantities, it follows indeed that for the (p,q)-$\mathcal{JC}m$ the action identity constraint is not equivalent to requiring an algebraic constraint on the ladder operators of the (p_0,q_0)-deformed Fock algebra type.

This choice also allows for the factorization of the Hamiltonian in (36) in the form

$$\mathbb{H}^{\rm red} = \mathbb{A}^+ \left(\frac{q^\mu}{p^\nu}\right)^{-2N} \mathbb{A} + \sum_{n=0,\pm}^{\infty} |n,\pm\rangle\, E^\pm_0\, \langle n,\pm|, \tag{92}$$

extending a similar expression in [14].

Given this solution for the factors $K^0_\pm([n])$, the general moment problem (68) reduces to the following conditions,

$$\int_0^{R^2} du\ h^\pm(u) = 1; \qquad \int_0^{R^2} du\, u^n\, h^\pm(u) = \prod_{k=1}^{n}\left(E^\pm_k - E^\pm_0\right), \quad n = 1,2,3,\cdots, \tag{93}$$

where the radius of convergence R is given as

$$R = \min\left(R_+, R_-\right), \qquad R_\pm = \lim_{n\to+\infty} \sqrt{E^\pm_n - E^\pm_0}. \tag{94}$$

In the absence of a detailed analysis of the energy spectra $E^\pm_n$ as functions of the parameters p, q, λ and ϵ and the function $h(p,q)$, nothing more explicit may be said concerning this moment problem. Since when $p > 1$ the quantities $[n]$ always possess a turn-around behaviour as functions of n for n sufficiently large, it is to be expected generally that the radius of convergence R, hence the moment problem as well, are associated to a finite disk D_R in the complex plane. Nevertheless, one conclusion of the present discussion is that indeed for the (p,q)-VCS considered in this work, the action identity constraint leads to coherent states different from those constructed in Section 5.1 and for which explicit solutions to the moment problem have been given.

5.3 The spin decoupled limit $\lambda = 0$

In the limit that $\lambda = 0$, the two spin sectors of the model are decoupled, and the (p,q)-$\mathcal{JC}m$ reduces to the supersymmetric harmonic oscillator [43, 44, 18] with a (p,q)-deformation. Diagonalization of the reduced Hamiltonian (24) is then of course straightforward in the σ_3-eigenbasis, with, for $n = 0, 1, 2, \cdots$,

$$\mathcal{H}^{\rm red}_{\lambda=0}\,|n,\pm\rangle = \epsilon^\pm_n\,|n,\pm\rangle, \qquad \epsilon^\pm_n = (1+\epsilon)h(p,q)[n] \,+\, \frac{1}{2}(1+\epsilon) \pm \frac{1}{2}. \tag{95}$$

From that point of view, one thus has two decoupled (p,q)-deformed Fock bases, for which one could consider the usual (p,q)-coherent states in each spin sector separately. However, such coherent states do not coincide with any of those constructed in this paper and obtained in the limit $\lambda = 0$, because of the distinguished role played by the singleton state $|E_*\rangle = |0,-\rangle$ and the S_2 unit vector character of the (p,q)-VCS. In particular the ladder operators $\mathcal{A}^\pm$ acting within each of the towers $|E^\pm_n\rangle$ do not coincide with the annihilation and creation operators a and $a^\dagger$ defining the Hamiltonian (24), even in the decoupled limit $\lambda = 0$. As a matter of fact, the action of the ladder operators $\mathcal{A}^\pm$ may switch between the two spin sectors as a function of n depending on the sign of the quantity $\mathcal{E}([n+1])$.

More specifically, let us introduce the notation

$$s_n = {\rm sign}\,\mathcal{E}([n+1]), \qquad n \in \mathbb{N}. \tag{96}$$

In the limit that $\lambda = 0$, one has $Q([n+1]) = |\mathcal{E}([n+1])|/2$, so that the mixing angle $\theta([n])$ is now such that, for all $n \in \mathbb{N}$,

$$\lambda = 0: \quad \sin\theta([n]) = \frac{1}{2}(1-s_n)\,({\rm sign}\,\lambda), \quad \cos\theta([n]) = \frac{1}{2}(1+s_n). \tag{97}$$

Consequently, the towers of energy eigenstates $|E^\pm_n\rangle$ are then given as follows, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\text{If } s_n = +1: \quad |E^+_n\rangle_{\lambda=0} = |n+1,-\rangle, \qquad |E^-_n\rangle_{\lambda=0} = |n,+\rangle; \\ &\text{If } s_n = -1: \quad |E^+_n\rangle_{\lambda=0} = ({\rm sign}\,\lambda)\,|n,+\rangle, \qquad |E^-_n\rangle_{\lambda=0} = -({\rm sign}\,\lambda)\,|n+1,-\rangle, \end{aligned} \tag{98}$$

while the energy eigenvalues are given as

$$\begin{aligned} \text{If } s_n = +1: \quad & E^+_n(\lambda=0) = (1+\epsilon)h(p,q)[n+1] \,+\, \tfrac{1}{2}(1+\epsilon) \,-\, \tfrac{1}{2}, \\ & E^-_n(\lambda=0) = (1+\epsilon)h(p,q)[n] \,+\, \tfrac{1}{2}(1+\epsilon) \,+\, \tfrac{1}{2}; \\ \text{If } s_n = -1: \quad & E^+_n(\lambda=0) = (1+\epsilon)h(p,q)[n] \,+\, \tfrac{1}{2}(1+\epsilon) \,+\, \tfrac{1}{2}, \\ & E^-_n(\lambda=0) = (1+\epsilon)h(p,q)[n+1] \,+\, \tfrac{1}{2}(1+\epsilon) \,-\, \tfrac{1}{2}. \end{aligned} \tag{99}$$

These spectra do indeed coincide with those in (95), once the singleton state $|E_*\rangle = |0,-\rangle$ with $E_* = \epsilon/2$ is included as well.

These expressions show how, even in the decoupled spin limit $\lambda = 0$, the (p,q)-VCS constructed here are not simply the juxtaposition of two separate (p,q)-coherent states of the (p,q)-deformed Fock algebra in each of the two spin sectors. Since the spectrum of the system is discrete infinite, by leaving aside the singleton state $|0,-\rangle$, all the remaining states still allow for similar types of constructions of coherent states, but in such a way that different spin sectors are getting superposed, leading to the SU(2) vector coherent states of the type studied here. All the expressions detailed in the previous sections for the (p,q)-VCS may readily be particularized to the limit $\lambda \to 0$.

6 Conclusion

In this work, we considered (p, q)-deformations of the Jaynes-Cummings model in the rotating wave approximation, extending recent developments on this topic in the non-deformed case [14]. Having introduced (p, q)-deformed versions of the model, first its energy eigenspectrum has been identified, enabling the definition of different relevant operators acting on Hilbert space and the characterization of the spectrum in terms of two separate infinite discrete towers and a singleton state. Among these operators, ladder operators acting within each of the two towers separately may be considered, defined up to some arbitrary normalization factors.

Such a structure sets the stage for the introduction of vector coherent states for the (p, q)-deformed Jaynes-Cummings model, following the approach of [14] and the rationale outlined in [30]. These (p, q)-VCS are parametrized by elements of $\mathbb{C} \times S_2$, and enjoy temporal stability through a further action-angle identification. The moment problem associated to the overcompleteness property of these (p, q)-VCS involves SU(2)-valued matrix weight functions. Using (p, q)-arithmetic techniques, some explicit and exact solutions to the moment problem have been displayed, hence characterizing specific classes of such (p, q)-VCS. All these solutions provide (p, q)-extensions to the non-deformed vector coherent states of the $\mathcal{JC}m$ considered in [14]. These explicit solutions are obtained by requiring that specific algebraic constraints of the (p, q)-deformed Fock algebra type be obeyed by the ladder operators. However, in contradistinction to [14], we have not been able to display an explicit and exact solution to the moment problem in the generic case by imposing an action identity constraint.

Finally, the spin decoupled limit of these models was considered, corresponding to a (p, q)-supersymmetric oscillator of which the two sectors are intertwined in a manner depending on the sign of the energy level spacing between the two decoupled spin sectors as function of the excitation level. In the non-deformed limit $(p, q) = (1, 1)$, this feature disappears, reproducing the ordinary supersymmetric oscillator. Our results thus provide new classes of generalized versions of the $\mathcal{JC}m$ in the rotating wave approximation [20, 18]. Finally, the (p, q)-VCS built here extend the q-coherent states obtained by other techniques involving supersymmetric shape invariance and self-similar potential formalisms applied to the harmonic oscillator [20, 45].

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Appendix

This appendix lists some useful facts related to the (p,q)-boson algebra and associated functions. The (p,q)-deformed oscillator algebra introduced in [5] is generated by operators a, $a^\dagger$ and N obeying the relations

$$\begin{aligned} [N,a] &= -a, \qquad & [N,a^\dagger] &= a^\dagger, \\ aa^\dagger - qa^\dagger a &= p^{-N}, \qquad & aa^\dagger - p^{-1}a^\dagger a &= q^N. \end{aligned} \tag{100}$$

Throughout the text, we assume the real parameters p and q are such that $p>1$, $0<q<1$ and $pq<1$. The limit $p\to 1^+$ yields the q-oscillator of Arik and Coon [3] while $p=q$ gives the q-deformed oscillator algebra of Biedenharn and MacFarlane [4]. Finally, the algebra (100) reduces to the ordinary harmonic oscillator Fock algebra as $q\to 1$ for $p=1^+$ or $p=q$. At any stage of the discussion, the (p,q)-deformed model readily reduces to its usual counterpart as $(p,q)\to(1,1)$.

The associated (p,q)-deformed Fock-Hilbert space representation is spanned by the vacuum $|0\rangle$ annihilated by a and the orthonormalized states $|n\rangle$, such that

$$\begin{aligned} & a|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad |n\rangle = \frac{1}{\sqrt{[n]_{(p,q)}!}}\left(a^\dagger\right)^n|0\rangle, \\ & a|n\rangle = \sqrt{[n]_{(p,q)}}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]_{(p,q)}}|n+1\rangle, \quad N|n\rangle = n|n\rangle, \end{aligned} \tag{101}$$

where the symbol $[n]_{(p,q)} = (p^{-n}-q^n)\,/\,(p^{-1}-q)$ is called (p,q)-basic number with, by convention, $[0]_{(p,q)}=0$, and its (p,q)-factorial is defined through $[n]_{(p,q)}! = [n]_{(p,q)}\left([n-1]_{(p,q)}!\right)$ and the convention $[0]_{(p,q)}!=1$. There exists a formal (p,q)-number operator denoted by $[N]_{(p,q)}$, or simply by $[N]$ when no confusion arises. As a matter of fact, from the second pair of relations in (100), it follows that $[N]=a^\dagger a$ as well as $[N+1]=aa^\dagger$. One has of course $[N]|n\rangle = [n]|n\rangle$. Hence, (101) provides a well defined Fock-Hilbert representation space of the algebra (100).

The following relations hold for any function $f\equiv f(N)$ and consequently for any function of $[N]$,

$$af(N-1) = f(N)a, \quad a^\dagger f(N) = f(N-1)a^\dagger. \tag{102}$$

Let us define q-shifted products and factorials and their (p,q)-analogues. Using the notations of [46], for any quantity x, $(x;q)_\alpha$ is constructed as follows,

$$(x;q)_0 = 1, \qquad (x;q)_\alpha = \frac{(x;q)_\infty}{(xq^\alpha;q)_\infty}, \qquad (x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)\,. \tag{103}$$

Furthermore, in the notations of [10], (p,q)-shifted products and factorials are defined as follows, for any real quantities a and b such that $a\neq 0$,

$$[a,b;p,q]_0 = 1, \qquad [a,b;p,q]_\alpha = \frac{[a,b;p,q]_\infty}{[ap^\alpha,bq^\alpha;p,q]_\infty}, \qquad [a,b;p,q]_\infty = \prod_{n=0}^{\infty}\left(\frac{1}{ap^n}-bq^n\right). \tag{104}$$

For $\alpha = n\in\mathbb{N}$, we have

$$[p^\mu,q^\nu;p,q]_n \;=\; \left(\frac{1}{p^\mu}-q^\nu\right)\left(\frac{1}{p^{\mu+1}}-q^{\nu+1}\right)\cdots\left(\frac{1}{p^{\mu+n-1}}-q^{\nu+n-1}\right)$$

$$= \quad p^{-\mu n - n(n-1)/2}(p^{\mu}q^{\nu}; pq)_n. \tag{105}$$

This identity is a central formula since it defines a bridge between q- and (p,q)-analogue quantities and functions.

Let us now introduce q-analogues of the ordinary exponential funtion. There exist many types of q-deformations of the exponential function e^z, $z \in \mathbb{C}$ (see, for instance, [9]). For any $(z,\mu) \in \mathbb{C} \times \mathbb{R}$, the (μ,q)-exponential is the complex function [9]

$$E_q^{(\mu)}(z) = \sum_{n=0}^{\infty} \frac{q^{\mu n^2}}{(q;q)_n} z^n. \tag{106}$$

This series has an infinite radius of convergence for $\mu > 0$. For $\mu = 0$ its domain of definition reduces to the unit disk, $|z| < 1$, while it is nowhere convergent in $\mathbb{C}$ for $\mu < 0$. Rescaling $z \to z(1-q)$ and taking the limit $\lim_{q\to 1} E_q^{\mu}(z(1-q))$, one recovers e^z. For some specific values of μ, (106) reproduces some standard q-exponentials [9, 11],

$$\begin{aligned} E_q^{(0)}(z) &= e_q(z) = \frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \qquad |z| < 1, & (107) \\ E_q^{(1/2)}(z) &= E_q(q^{1/2}z) = (-q^{1/2}z;q)_\infty, \qquad z \in \mathbb{C}, & (108) \end{aligned}$$

where

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{(q;q)_n}, \qquad z \in \mathbb{C}, \tag{109}$$

is known as the Jackson q-exponential [6]. Note that whereas $E_q^{(\mu)}(z)$ is defined in the entire complex plane, $|z| < \infty$, for any $\mu > 0$, its reduction $e_q(z)$ is only defined on the unit disc. Finally, it is also well established that [11]

$$E_q(-z)e_q(z) = 1. \tag{110}$$

(p,q)-analogues of the usual exponential function e^z, $z \in \mathbb{C}$ may also be introduced (see, for instance, [10]). Given any $(z,\mu,\nu) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}$, consider the (μ,ν,p,q)-exponential function

$$\mathcal{E}_{(p,q)}^{(\mu,\nu)}(z) = \sum_{n=0}^{\infty} \left(\frac{q^{\mu}}{p^{\nu}}\right)^{n^2} \frac{z^n}{[p,q;p,q]_n}. \tag{111}$$

Keeping in mind the condition $pq < 1$, the radius of convergence R of this series is such that

$$R_1 = \begin{cases} \infty, & \text{if } q^{2\mu}p^{1-2\nu} < 1; \\ p^{\nu-1}q^{-\mu}, & \text{if } q^{2\mu}p^{1-2\nu} = 1; \\ 0, & \text{if } q^{2\mu}p^{1-2\nu} > 1. \end{cases} \tag{112}$$

Thus the function $\mathcal{E}_{(p,q)}^{(\mu,\nu)}(z)$ exists only provided $q^{2\mu}p^{1-2\nu} \le 1$.

In order to recover the usual exponential function, one has to rescale $z \to z(p^{-1} - q)$, for example, and then take the limit $\lim_{(p,q)\to(1,1)} \mathcal{E}_{(p,q)}^{\mu,\nu}(z(p^{-1}-q)) = e^z$. For particular values of the parameters μ and ν, (111) reproduces known (p,q)-exponentials,

$$\mathcal{E}_{(p,q)}^{(1/2,1/2)}(z) = E_{(p,q)}\left(\left(\frac{q}{p}\right)^{1/2} z\right) = \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^{n^2/2} \frac{z^n}{[p,q;p,q]_n}, \tag{113}$$

where

$$E_{(p,q)}(z) \;=\; \sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n(n-1)/2}\frac{z^n}{[p,q;p,q]_n}. \tag{114}$$

The function $E_{(p,q)}$ may be found in [10]. Note that (114) coincides with (109) as $p \to 1$. In the same limit, (111) reproduces the (μ, q)-deformed exponential map $E_q^{(\mu)}(z)$ [9]. If $\mu = 0 = \nu$ the series (111) is not defined since then $R = 0$, unless one has taken $p = 1$ in which case the radius of convergence is unity. A (p, q)-analogue of (107) is given by

$$e_{(p,q)}(z) = \sum_{n=0}^{\infty}\frac{1}{p^{n^2/2}}\frac{z^n}{[p,q;p,q]_n}, \qquad |z| < p^{-1/2}, \tag{115}$$

which reproduces exactly $e_q(z)$ converging in the unit disc as $p \to 1^+$. Furthermore, we have from (105)

$$e_{(p,q)}(z) = \sum_{n=0}^{\infty}\frac{(p^{1/2}z)^n}{(pq;pq)_n} = e_{pq}(p^{1/2}z). \tag{116}$$

Using (105) and (109), we may also write

$$\begin{aligned} E_{(p,q)}(z) \;&=\; \sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n(n-1)/2}\frac{z^n}{p^{-n(n+1)/2}(pq;pq)_n} \\ &=\; \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{(zp)^n}{(pq;pq)_n} = E_{pq}(pz). \end{aligned} \tag{117}$$

Then taking into account (110), (116) and (117), a (p, q)-analogue of (110) is given by

$$E_{pq}(-pz)e_{pq}(pz) = E_{(p,q)}(-z)e_{(p,q)}(p^{1/2}z) = 1. \tag{118}$$

Finally, consider

$$\mathfrak{e}_{(p,q)}^{(\mu,\nu)}(z) = \sum_{n=0}^{\infty}\left(\frac{q^\mu}{p^\nu}\right)^{n^2}\frac{z^n}{n!}. \tag{119}$$

Therefore, $\mathfrak{e}_{(p,q)}^{(\mu,\nu)}(z)$, which converges to e^z as $(p, q) \to (1, 1)$, provides a (p, q)-deformed exponential analogue to the q-function used by Penson and Solomon [42] which coincides with $\mathfrak{e}_{(1,q)}^{(1,\nu)}(q^{-1/2}z)$. The radius of convergence of (119) is given as

$$R_2 = \begin{cases} \infty, & \text{if } q^\mu p^{-\nu} \leq 1; \\ 0, & \text{if } q^\mu p^{-\nu} > 1. \end{cases} \tag{120}$$

Finally, consider the Ramanujan integral [7, 19], valid for any integer $n \in \mathbb{N}$,

$$\int_0^\infty dt\, t^n\, e_q(-t) = -\frac{(q;q)_n}{q^{n(n+1)/2}}\log q. \tag{121}$$

Through the change of variables

$$q \to pq, \qquad t \to \lambda_0\, p^{-1/2}\, t, \qquad \lambda_0 > 0, \tag{122}$$

and using once again (105), the following identity is obtained, for any $n \in \mathbb{N}$,

$$\int_0^\infty dt\, t^n\, e_{(p,q)}\left(-\lambda_0 p^{-1/2} t\right) = \frac{[p,q;p,q]_n}{\lambda_0^{n+1}\, q^{n(n+1)/2}}\log\left(\frac{1}{pq}\right). \tag{123}$$

This result is indeed a (p, q)-analogue of the Ramanujan integral (121).

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